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THERMAL INSTABILITY OF HARTMANN FLOW IN THE THERMAL ENTRANCE REGION OF HORIZONTAL PARALLEL-PLATE CHANNELS HEATED FROM BELOW

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Abstract—The onset of instability in the form of longitudinal vortices for fully developed Hartmann laminar flow in the thermal entrance region of horizontal parallel-plate channels is investigated by a numerical method for the case with a uniform vertical magnetic field and heating from below. Numerical results are obtained for $Pr = 0.7, 0.01$, $Pe = 10\,100, \infty$, $Br = 0, -1$ and $Ha = 0, 2, 6, 10$. The effects of Prandtl, Peclet (axial conduction), Brinkman (viscous dissipation and Joule heating) and Hartmann numbers on thermal instability of magneto-hydrodynamic flow are studied.

NOMENCLATURE

a, dimensionless wave number;
B, magnetic field induction vector, (O, O, B_0);
Br, Brinkman number, $\mu_f U_m^2/(k\theta_c)$;
b, dimensionless perturbation vector of impressed magnetic field, (b_x, b_y, b_z);
C_n, D_n, coefficients in the series expansion of θ_c;
c_p, specific heat at constant pressure;
D, d/dz;
E, electric field intensity vector, (O, E_0, O);
E_n, O_n, even and odd eigenfunctions;
e, dimensionless perturbation vector of electric field, (e_x, e_y, e_z);
Gr, Grashof number, $g\beta(\Delta T)l^3/\nu^2$;
g, gravitational acceleration;
Ha, Hartmann number, $(\sigma/\mu_f)^{1/2}B_0 l$;
J, electric current density vector, (O, J_y, O);
J_0, $\sigma B_0 U_m J$;
j, dimensionless perturbation vector of **J**, (j_x, j_y, j_z);
K, external loading parameter, $E_0/(B_0 U_m)$;
k, thermal conductivity;
L, a distance between two infinite horizontal flat plates;
l, $L/2$;
M, number of divisions in y direction;
P, P_b, fluid pressure $(P_b + P')$ and pressure for basic flow;
Pe, Peclet number, $PrRe$;
Pr, Prandtl number, $c_p\mu_f/k$;
p, dimensionless perturbation pressure, $P'/(\rho U_m^2)$;
Ra, Rayleigh number, $g\beta(\Delta T l^3/\nu\alpha$;
Re, Reynolds number, $\rho U_m l/\mu_f$;
Rm, magnetic Reynolds number, $U_m l/(1/\mu_e\sigma)$;
T, T_b, T_0, fluid temperature $(T_b + \theta')$, fluid temperature of basic flow and uniform entrance temperature;
T_1, T_2, T_m, uniform but different lower and upper plate temperatures, and $(T_1 + T_2)/2$;
U_b, U_m, u_b, axial, mean and dimensionless velocities of basic flow;
u, v, w, dimensionless perturbation velocity components;
$\mathbf{V}, \mathbf{V}_b, \mathbf{V}'$, velocity vector $(\mathbf{V}_b + \mathbf{V}')$, basic velocity vector (U_b, O, O) and perturbation velocity vector (U', V', W');
X, Y, Z, Cartesian coordinates with origin at lower plate;
x, y, z, dimensionless coordinates;
$\mathbf{x}, \mathbf{z}$, transformed coordinates, x/Pe, $\mathbf{z} = z$;
Z', z', dimensional and dimensionless transverse coordinates with origin at center of channel.

Greek symbols

α, thermal diffusivity;
β, coefficient of thermal expansion;
β_n, γ_n, even and odd eigenvalues;
$\theta, \theta_b, \theta_0$, dimensionless perturbation, basic flow and entrance temperatures;
$\theta_c, \theta_e, \theta_f$, characteristic temperature difference $(T_2 - T_m) = (T_2 - T_1)/2$, and dimensionless fluid temperatures defined by equation (7);
μ_e, μ_f, magnetic permeability and viscosity of fluid;
ν, kinematic viscosity;
ρ, fluid density;
σ, electric conductivity;
Φ, viscous dissipation function;
ϕ_u, ϕ_θ, dimensionless basic velocity and temperature profiles, U_b/U_m and $(T_b - T_2)/\Delta T$;
Ψ, dimensionless stream function;
ΔT, $(T_1 - T_2) = -2\theta_c$.

Superscripts and subscripts

$'$, perturbation quantity;
$+$, amplitude of disturbance quantity;
$*$, transformed perturbation variable or critical value;
b, basic quantity in unperturbed state.

1. INTRODUCTION

IN RECENT years, the problem of the laminar forced convection for fully developed MHD laminar flow in the thermal entrance region of a parallel-plate channel has been studied by many investigators for the thermal boundary conditions of both uniform wall heat flux and constant wall temperature. The literature on the subject is well reviewed in [1, 2] and further recent works are quoted in [3]. It is known that when a horizontal fluid layer is subjected to an adverse temperature gradient, a top-heavy situation results and the system is potentially unstable due to the buoyancy forces. With a superposed fully developed laminar flow between two horizontal flat plates, heated from below, the onset of the secondary flow in the form of longitudinal vortices [4–8] is characterized by a critical Rayleigh number. With the appearance of the vortex rolls, the flow takes on a three-dimensional character and the heat transfer rate is expected to increase with the Rayleigh number. Thus, it is of practical interest to determine the conditions for the onset of secondary flow.

The effects of a vertical, uniform magnetic field on the thermal instability of horizontal stationary fluid layers were studied theoretically by Thompson [9] and Chandrasekhar [10, 11] and experimentally by Nakagawa [12–15]. The thermal instability of a magnetofluid in a vertical rectangular channel heated from below was investigated by Yu [16] quoting the related references. The thermal instability of a Hartmann flow in the thermal entrance region of a horizontal parallel-plate channel with heating from below does not appear to have been studied in the past. The purpose of this study is to determine the conditions marking the onset of longitudinal vortex rolls in the said passage where the two plates are maintained at uniform but different surface temperatures. The present study can be regarded as a first step toward investigating the change of heat-transfer rate due to the thermal instability for a Hartmann flow and represents an extension of the thermal instability problem for a confined horizontal fluid layer studied by Thompson [9] and Chandrasekhar [10, 11] to the case with a superposed fully developed laminar flow. The basic velocity and temperature fields in the thermal entrance region of the channel required for the present thermal instability analysis are reported in [3]. For the basic flow and temperature fields, the free convection effect is neglected and the problem is to find the condition at which free convection starts to affect the Hartmann flow.

2. FORMULATION OF THE THERMAL INSTABILITY PROBLEM

2.1. *Basic flow and temperature fields*

Consideration is given to a Hartmann flow between two horizontal flat plates under the action of a homogeneous transverse magnetic field B_0 and heated from below. The basic equations of motion, of Maxwell, and of energy appropriate to the thermal entrance region heat-transfer problem [3] are:

$$\nabla\cdot\mathbf{V}_b = 0 \tag{1}$$

$$(\mathbf{V}_b\cdot\nabla)\mathbf{V}_b = -\frac{1}{\rho}\nabla P_b + \nu\nabla^2\mathbf{V}_b + \frac{1}{\rho}\mathbf{J}\times\mathbf{B} \tag{2}$$

$$\begin{gathered}\nabla\cdot\mathbf{B} = 0,\quad \nabla X\mathbf{B} = \mu_e\mathbf{J},\quad \nabla\cdot\mathbf{E} = 0,\\ \nabla X\mathbf{E} = 0,\quad \mathbf{J} = \sigma(\mathbf{E}+\mathbf{V}_b X\mathbf{B})\end{gathered} \tag{3}$$

$$\rho c_p(\mathbf{V}_b\cdot\nabla)T_b = k\nabla^2 T_b + \Phi + \frac{1}{\sigma}(\mathbf{J}\cdot\mathbf{J}) \tag{4}$$

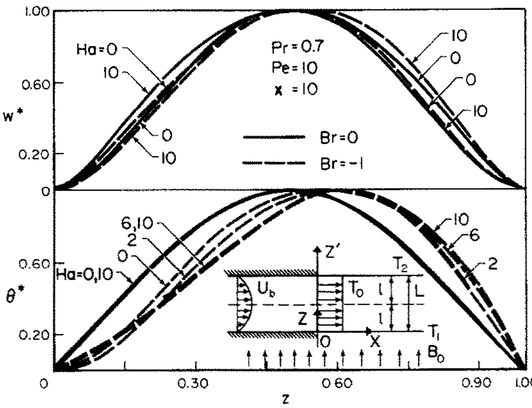

FIG. 1. Disturbance profiles for perturbation amplitudes w^* and θ^* at $Ha = 0$, 10 for $Pr = 0.7$, $Pe = 10$, $Br = 0$, -1 and $x = 10$.

where $\mathbf{V}_b = (U_b, O, O)$, $\mathbf{J} = [O, J_y = \sigma(E_0 - U_b B_0), O]$, $\mathbf{E} = (O, E_0, O)$, $\mathbf{B} = (O, O, B_0)$, $\Phi = \mu_f(\mathrm{d}U_b/\mathrm{d}Z')^2$ and $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Z'^2$ in energy equation (4) and the coordinate system is defined in Fig. 1. The boundary conditions are:

$$\begin{gathered}U_b(O, \pm l) = 0,\quad T_b(O, Z') = T_0,\\ T_b(X, -l) = T_1,\quad T_b(X, l) = T_2.\end{gathered} \tag{5}$$

Introducing the following dimensionless variables and physical parameters,

$$\begin{gathered}(X, Z') = [l](xPe, z'),\ U_b = [U_m](U_b),\\ \theta_b = (T_b - T_m)/(T_2 - T_m),\ \theta_0 = (T_0 - T_m)/(T_2 - T_m),\\ Re = \rho U_m l/\mu_f,\ Pe = PrRe = \rho c_p U_m l/k,\\ Ha = (\sigma/\mu_f)^{1/2}B_0 l,\ K = E_0/(B_0 U_m),\ Br = \mu_f U_m^2/(k\theta_c),\end{gathered}$$

where

$$\begin{gathered}U_m = \int_{-l}^{l} U_b\,\mathrm{d}Z'/(2l),\ T_m = (T_1 + T_2)/2,\\ \theta_c = T_2 - T_m = (T_2 - T_1)/2,\end{gathered}$$

the well known Hartmann solution [17] for equation (2) and the solution [3] of energy equation (4) considering both the viscous dissipation and axial conduction effects can be written as

$$\begin{aligned}u_b &= Ha(\cosh Ha - \cosh Haz')/(Ha\cosh Ha - \sinh Ha)\\ &= C_1(\cosh Ha - \cosh Haz')\end{aligned} \tag{6}$$

$$\theta_b = \theta_f(z') + \theta_e(x, z') \tag{7}$$

where

$$\theta_f = z' + Br[(C_1^2/4)(\cosh 2Ha - \cosh 2Haz') + 2C_1 C_2(\cosh Ha - \cosh Haz) + (C_2^2 Ha^2/2)(1-z'^2)],$$

$$C_2 = K - C_1 \cosh Ha,$$

$$\theta_e = \sum_{n=1}^{\infty} C_n E_n(z')\exp(-\beta_n x) + \sum_{n=1}^{\infty} D_n O_n(z')\exp(-\gamma_n x).$$

The details of the infinite series solution for θ are given in [3] and the expression for θ is given here for reference purpose only. At this point, it is convenient to shift the coordinate origin to the bottom plate for the instability problem and one obtains $z = \frac{1}{2}(z'+1)$ and the developing temperature profile $\phi_\theta = \frac{1}{2}(1-\theta_b)$.

2.2. *Perturbation equations*

In order to study the thermal instability concerned with the onset of secondary flow in the form of longitudinal vortices for the horizontal Hartmann flow heated from below, the perturbation quantities are superimposed on the basic quantities as

$$\begin{aligned} &\mathbf{V} = \mathbf{V}_b + \mathbf{V}' = [U_b(Z) + U', V', W'],\ T = T_b + \theta' \\ &P = P_b + P',\ \mathbf{E} = \mathbf{E}_b + \mathbf{e}' = (e'_x, E_0 + e'_y, e'_z), \\ &\mathbf{B} = \mathbf{B}_b + \mathbf{b}' = (b'_x, b'_y, B_0 + b'_z) \\ &\mathbf{J} = \mathbf{J}_b + \mathbf{j}' = (j'_x, J_0 + j'_y, j'_z). \end{aligned} \tag{8}$$

The above perturbation quantities are considered to be in the steady state and are of a function of space variables X, Y and Z only. After applying the linear stability theory and using Boussinesq approximation, the perturbation equations become:

$$\frac{\partial U'}{\partial X} + \frac{\partial V'}{\partial Y} + \frac{\partial W'}{\partial Z} = 0 \tag{9}$$

$$\rho\left(U_b \frac{\partial U'}{\partial X} + W' \frac{dU_b}{dZ}\right) = -\frac{\partial P'}{\partial X} + \mu_f \nabla^2 U' + (J_0 b'_z + B_0 j'_y) \tag{10}$$

$$\rho U_b \frac{\partial V'}{\partial X} = -\frac{\partial P'}{\partial Y} + \mu_f \nabla^2 V' - B_0 j'_x \tag{11}$$

$$\rho U_b \frac{\partial W'}{\partial X} = -\frac{\partial P'}{\partial Z} + \mu_f \nabla^2 W' - J_0 b'_x + \rho \mathbf{g} \beta \theta' \tag{12}$$

$$\rho c_p\left(U_b \frac{\partial \theta'}{\partial X} + U' \frac{\partial T_b}{\partial X} + W' \frac{\partial T_b}{\partial Z}\right) = k\nabla^2\theta' + 2\mu_f\left(\frac{dU_b}{dZ}\right)\left(\frac{\partial U'}{\partial Z}\right) + 2\frac{J_0}{\sigma} j'_y \tag{13}$$

$$j'_x = \sigma(e'_x + B_0 V'),\ j'_y = \sigma(e'_y - B_0 U' - U_b b'_z),\ j'_z = \sigma(e'_z + U_b b'_y) \tag{14}$$

$$\nabla \cdot \mathbf{e}' = 0,\ \nabla X \mathbf{e}' = 0,\ \nabla \cdot \mathbf{b}' = 0,\ \nabla X \mathbf{b}' = \mu_e \mathbf{j}' \tag{15}$$

where $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2 + \partial^2/\partial Z^2$.

Introducing the following nondimensional quantities and physical parameters,

$$(X, Y, Z) = L(x, y, z),\ (U', V', W') = U_m(u, v, w),$$
$$\theta' = (\Delta T)\theta,\ P' = (\rho U_m^2)p,\ (b'_x, b'_y, b'_z) = B_0(b_x, b_y, b_z),$$
$$(e'_x, e'_y, e'_z) = E_0(e_x, e_y, e_z),\ (j'_x, j'_y, j'_z) = \sigma B_0 U_m(j_x, j_y, j_z),$$
$$Gr = \frac{\mathbf{g}\beta(\Delta T)l^3}{\nu^2},\ Rm = U_m l/(1/\mu_e \sigma)$$

and noting that

$$U_b = U_m \phi_u,\ T_b - T_2 = (\Delta T)\phi_\theta,$$
$$J_0 = \sigma B_0 U_m J = \sigma B_0 U_m (K - \phi_u),$$
$$\Delta T = (T_1 - T_2) = -2\theta_c,$$

the perturbation equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{16}$$

$$\phi_u \frac{\partial u}{\partial x} + w\frac{d\phi_u}{dz} = -\frac{\partial p}{\partial x} + \frac{1}{2Re}\nabla^2 u + \frac{2Ha^2}{Re}(Jb_z + j_y) \tag{17}$$

$$\phi_u \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{2Re}\nabla^2 v - \frac{2Ha^2}{Re} j_x \tag{18}$$

$$\phi_u \frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{2Re}\nabla^2 w - \frac{2Ha^2}{Re} Jb_x + \frac{2Gr}{Re^2}\theta \tag{19}$$

$$2Pe\left[\phi_u \frac{\partial \theta}{\partial x} + u\frac{\partial \phi_\theta}{\partial x} + w\frac{\partial \phi_\theta}{\partial z}\right] = \nabla^2\theta - Br\left[\frac{d\phi_u}{dz}\frac{du}{dz} + 4Ha^2 J j_y\right] \tag{20}$$

$$j_x = Ke_x + v,\ j_y = Ke_y - u - \phi_u b_z,\ j_z = Ke_z + \phi_u b_y \tag{21}$$

$$\nabla \cdot \mathbf{e} = 0\ (a),\ \nabla X \mathbf{e} = 0\ (b),\ \nabla \cdot \mathbf{b} = 0\ (c),\ \nabla X \mathbf{b} = 2Rm\mathbf{j}\ (d). \tag{22}$$

Here it is understood that the operators ∇^2 and ∇ are dimensionless.

After eliminating u, v, p and using continuity equation, the three momentum equations can be combined into a single equation as

$$\nabla^2\nabla^2 w - 4Ha^2\frac{\partial^2 w}{\partial z^2} - 2Re\left[\phi_u \frac{\partial}{\partial x}\nabla^2 w - \frac{\partial w}{\partial x}\frac{d^2\phi_u}{dz^2}\right] = -\frac{4Gr}{Re}\nabla_1^2\theta + 4Ha^2\left[J\left(\nabla_1^2 b_x + \frac{\partial^2 b_z}{\partial x \partial z}\right) - 2\frac{d\phi_u}{dz}\frac{\partial b_z}{\partial x} - \phi_u \frac{\partial^2 b_z}{\partial x \partial z}\right] \tag{23}$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

From equations (20), (23) and noting further that for vortex-type instability $\partial p/\partial x = 0$, one has 7 unknowns u, w, θ, b_x, b_z, j_y and e_y. Consequently, one needs additionally one momentum equation, Ohm's law, two magnetic induction equations and one electric field equation as follows.

$$\nabla^2 u - 2Re\phi_u \frac{\partial u}{\partial x} = 2Rew\frac{d\phi_u}{dz} - 4Ha^2[(K - \phi_u)b_z + j_y] \tag{24}$$

$$j_y = Ke_y - u - \phi_u b_z \tag{25}$$

$$\nabla^2 b_x - 2Rm\phi_u \frac{\partial b_x}{\partial x} = -2Rm\left[\frac{\partial u}{\partial z} + \frac{d\phi_u}{dz} b_z\right] \tag{26}$$

$$\nabla^2 b_z - 2Rm\phi_u \frac{\partial b_z}{\partial x} = -2Rm\frac{\partial w}{\partial z} \tag{27}$$

$$\nabla^2 e_y = 0. \tag{28}$$

The boundary conditions are

$$u = w = \frac{\partial w}{\partial z} = \theta = 0 \quad \text{at} \quad z = 0, 1 \quad \text{(rigid walls)}$$

$$j_z = 0 \quad \text{at} \quad z = 0, 1 \quad \text{(non-conducting walls).} \qquad (29)$$

For the disturbances in the form of stationary longitudinal vortices, one may assume the disturbance form [7] $f = f^{+}(z)\,e^{iay}$ for the disturbance quantities. The assumption of taking the perturbation quantities to be "in the steady state" is consistent with the previous investigations [7, 8, 19]. The set of equations then becomes:

$$(D^2-a^2)^2 w^+ - 4Ha^2 D^2 w^+ = 4Ha^2[(K-\phi_u)(-a^2)b_x^+] + \frac{4Gr}{Re}a^2\theta^+ \qquad (30)$$

$$(D^2-a^2)u^+ - 4Ha^2 u^+ = 2Re\frac{\mathrm{d}\phi_u}{\mathrm{d}z}w^+ - 4Ha^2[(K-2\phi_u)b_z^+] \qquad (31)$$

$$(D^2-a^2)\theta^+ = 2Pe\left[u^+\frac{\partial\phi_\theta}{\partial x} + w^+\frac{\partial\phi_\theta}{\partial z}\right] + Br\left[\frac{\mathrm{d}\phi_u}{\mathrm{d}z}Du^+ - 4Ha^2(K-\phi_u)j_y^+\right] \qquad (32)$$

$$j_y^+ = Ke_y^+ - u - \phi_u b_z^+ \qquad (33)$$

$$(D^2-a^2)b_x^+ = -2Rm\left(Du^+ + \frac{\mathrm{d}\phi_u}{\mathrm{d}z}b_z^+\right) \qquad (34)$$

$$(D^2-a^2)b_z^+ = -2RmDw^+ \qquad (35)$$

$$(D^2-a^2)e_y^+ = 0 \qquad (36)$$

where $D = \mathrm{d}/\mathrm{d}z$, $\nabla^2 = D^2-a^2$ and $\nabla_1^2 = -a^2$. A study of the electromagnetic boundary conditions is now in order. When the magnetic Reynolds number Rm is very small, an order of magnitude analysis reveals that the R.H.S. of equations (34) and (35) can be neglected. From Maxwell's equations and equations (34), (36) and (29) it can be shown that $e_x^+ = e_y^+ = e_z^+ = b_x^+ = 0$ for the whole domain and the details are given in [18]. It is convenient to introduce the transformations $x = Pe\mathbf{x}$, $z = \mathbf{z}$, $u^+ = Reu^*$, $w^+ = w^*$, $\theta^+ = Pe\theta^*$, $j_y^+ = Rej_y^*$, $b_z^+ = Rmb_z^*$ and one obtains

$$[(D^2-a^2)^2 - 4Ha^2D^2]w^* = 4Raa^2\theta^* \qquad (37)$$

$$[(D^2-a^2) - 4Ha^2]u^* = 2\frac{\mathrm{d}\phi_u}{\mathrm{d}z}w^* - \left[4Ha^2(K-2\phi_u)\frac{Rm}{Re}b_z^*\right] \qquad (38)$$

$$(D^2-a^2)\theta^* = 2\left[\frac{u^*}{Pr}\frac{\partial\phi_\theta}{\partial\mathbf{x}} + w^*\frac{\partial\phi_\theta}{\partial\mathbf{z}}\right] + \frac{Br}{Pr}\left[Re\frac{\mathrm{d}\phi_u}{\mathrm{d}\mathbf{z}}Du^* - 4Ha^2(K-\phi_u)j_y^*\right] \qquad (39)$$

$$j_y^* = -u^* - \left[\phi_u\frac{Rm}{Re}b_z^*\right]. \qquad (40)$$

Since Rm/Re is very small, the terms involving Rm/Re in equations (38) and (40) can be neglected [19] entirely in comparison with the other terms. Thus, one sees that the present eigenvalue problem can be solved independently of the boundary conditions on the magnetic field. The physical parameters are seen to be Pr, Pe, Br, Ha and Ra. The boundary conditions are

$$u^* = w^* = Dw^* = \theta^* = 0 \quad \text{at} \quad \mathbf{z} = 0, 1. \qquad (41)$$

It is instructive to identify the physical meaning of each term in the perturbation equations and regard the terms on the R.H.S. of each equation as the source or forcing terms. One also notes that the terms involving Ha^2 are preceded by a negative sign suggesting that the transverse magnetic field has a stabilizing effect on the instability. Without the effects of a magnetic field, Joule heating and viscous dissipation, the present thermal instability problem reduces to that studied in [7]. For given values of Pr, Pe, Br and Ha, one is interested in determining the minimum critical Rayleigh number and the corresponding wave number for the onset of instability as stationary longitudinal vortices through the solution of equations (37)–(41).

3. NUMERICAL SOLUTION

In view of the expressions for the basic velocity and temperature profiles, an analytical solution of the characteristic value problem is apparently not practical. A finite-difference method using an iterative technique is used for the simultaneous solution of the disturbance equations [7]. Using the higher order finite-difference scheme due to Thomas [20], equation (37) and its boundary conditions may be transformed into a quidiagonal system of matrix for a set of algebraic equations and two tridiagonal systems result from equations (38) and (39) and their boundary conditions. Noting that for given values of Pr, Pe, Br and Ha, the basic profiles ϕ_u and ϕ_θ are known, the solution of a coupled set of equations (37)–(41) can be carried out by using an iterative procedure. It is found that only a few iterations are required to determine critical Ra values to five significant figures. The complete details of the numerical solution as well as the method of finding the minimum critical Ra as a function of wave number a are given in [18].

4. RESULTS AND DISCUSSION

Before presenting the numerical results, it is well to note that the basic fully-developed velocity profile ϕ_u depends on Ha only and the basic temperature profile ϕ_θ is a function of the parameters Pe, Br and Ha and is independent of Pr. The typical profiles for ϕ_θ are shown in [3]. In the perturbation equations (37)–(39), only two prescribed parameters Pr and Ha appear. The numerical results will be presented in such a way to illustrate the effects of the aforementioned physical parameters on thermal instability.

The effects of the Hartmann number on disturbance profiles w^*, θ^* and u^* are shown in Figs. 1 and 2, respectively, for fully developed condition ($\mathbf{x} = 10$) with $Pr = 0.7$, $Pe = 10$ and $Br = 0$, -1. From the normal modes of the disturbances and the definition of the stream function $v = \partial\Psi/\partial z$, $w = -\partial\Psi/\partial y$, one obtains $\Psi = (iw^+/a)e^{iay}$ and one may compute the stream function Ψ by noting that physical meaning is attached

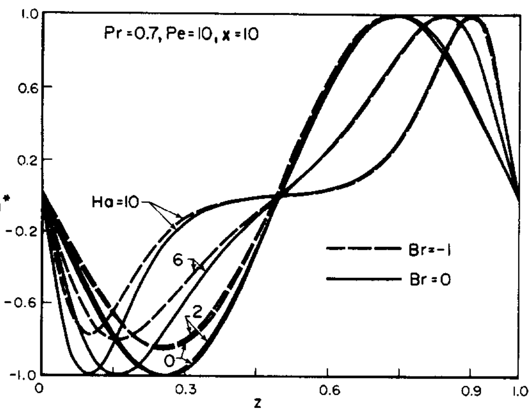

FIG. 2. Disturbance profiles for perturbation amplitudes u^* at $Ha = 0, 2, 6, 10$ for $Pr = 0.7$, $Pe = 10$, $Br = 0, -1$ and $\mathbf{x} = 10$.

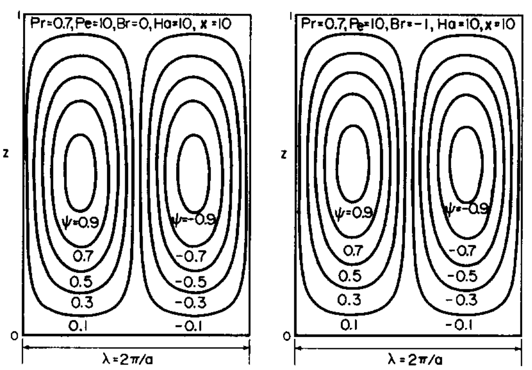

FIG. 3. Streamline pattern at onset of instability for $Pr = 0.7$, $Pe = 10$, $Ha = 10$, $\mathbf{x} = 10$, $Br = 0$ and -1.

only to the real part. The results are shown in Fig. 3. In Figs. 1–3, the magnitude of the maximum disturbance quantity is taken to be one. The neutral stability curves for $Pe = 10$, $Pr = 0.01$ and 0.7 are shown in Figs. 4 and 5, respectively, where one may see the effects of Hartmann and Brinkman numbers clearly.

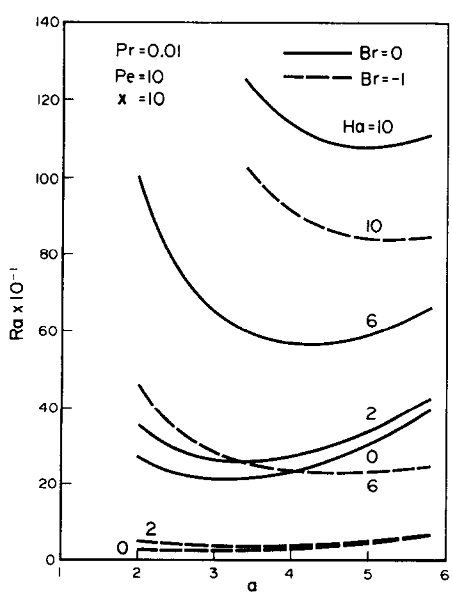

FIG. 4. Neutral stability curves for $Pr = 0.01$, $Br = 0, -1$ and $Ha = 0, 2, 6, 10$.

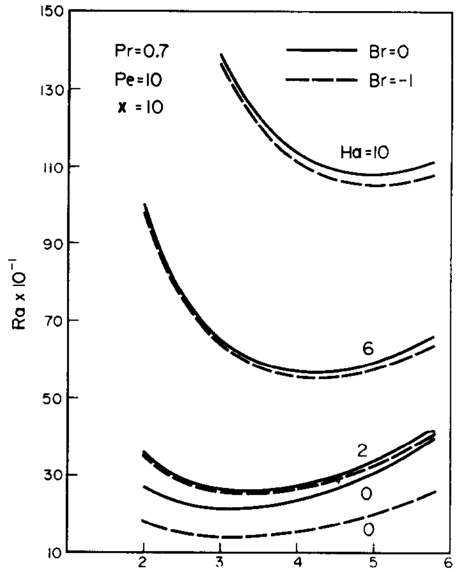

FIG. 5. Neutral stability curves for $Pr = 0.7$, $Br = 0, -1$ and $Ha = 0, 2, 6, 10$.

The effect of Peclet number on critical Rayleigh numbers Ra^* along the axial coordinate $\mathbf{x}$ is shown in Figs. 6–9 with $Pr = 0.01, 0.7$ and $Br = 0, -1$ for $Ha = 0, 2, 6$ and 10. In Fig. 6 ($Ha = 0$) with $Pr = 0.7$, the critical Ra^* is seen to decrease monotonically with $\mathbf{x}$ until an asymptotic value is approached. On the other hand, with $Pr = 0.01$ and $Br = -1$, a local maximum value for Ra^* exists at a certain axial location before reaching the asymptotic value. Furthermore, the region near the thermal entrance ($\mathbf{x} = 0$) is seen to be more unstable than the region near the fully developed region ($\mathbf{x} \gtrsim 10$) for $Pr = 0.01$. It is found that the curve for $Pe = 100$ can be regarded as $Pe = \infty$ practically. The merging of the two curves for $Pe = 10$ and 100 at some axial position signifies the disappearance of the axial conduction effect. With $Ha = Br = 0$, the asymptotic value of $Ra^* = 213.47$ which is independent of Prandtl number agrees with the well-known value of 1708/8 for the Benard problem. This can be explained from the perturbation equations. For fully developed flow, $\partial \phi_0/\partial x = 0$ in equation (39) and

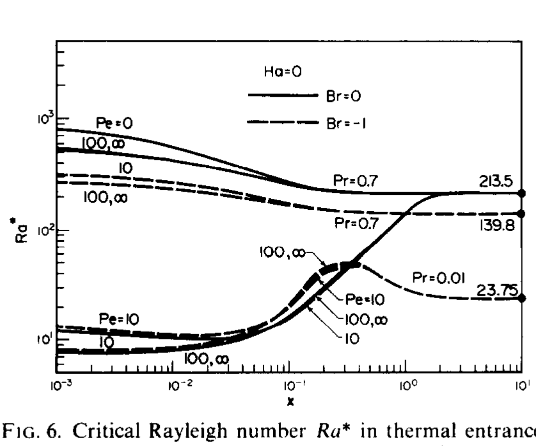

FIG. 6. Critical Rayleigh number Ra^* in thermal entrance region for $Pr = 0.01, 0.7$ and $Pe = 10, 100, \infty$ with $Ha = 0$, $Br = 0, -1$.

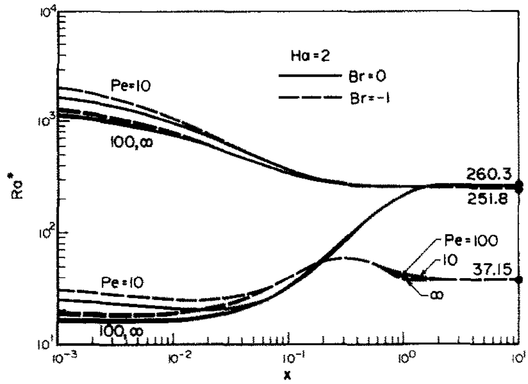

FIG. 7. Critical Rayleigh number Ra^* in thermal entrance region for $Pr = 0.01, 0.7$ and $Pe = 10, 100, \infty$ with $Ha = 2$, $Br = 0$ and -1.

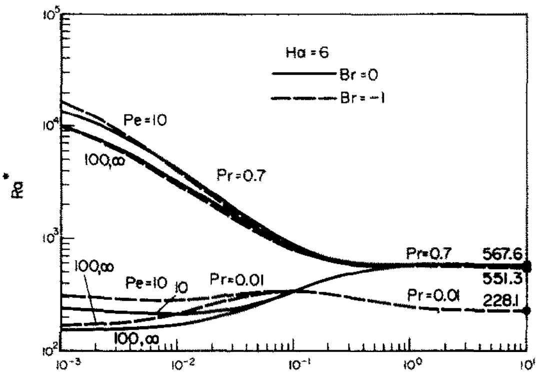

FIG. 8. Critical Rayleigh number Ra^* in thermal entrance region for $Pr = 0.01, 0.7$ and $Pe = 10, 100, \infty$ with $Ha = 6$, $Br = 0$ and -1.

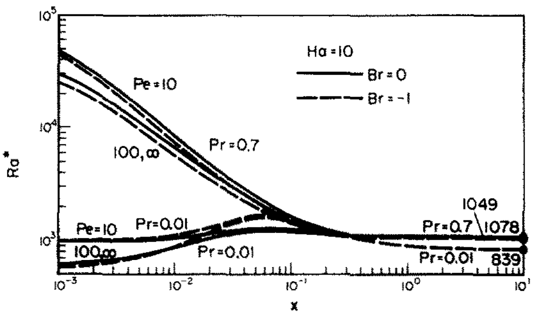

FIG. 9. Critical Rayleigh number Ra^* in thermal entrance region for $Pr = 0.01, 0.7$ and $Pe = 10, 100, \infty$ with $Ha = 10$, $Br = 0$ and -1.

with $Ha = Br = 0$, equations (37) and (39) become identical with those of the Benard problem.

It is difficult to explain the reasons for the occurrence of the local maximum for Ra^* in the thermal entrance region as noted earlier. Considering the case with $Ha = 0$, it appears that the cause for the phenomenon is due to the combined effect of the convective term $(u^*/Pr)\partial\phi_0/\partial x$ and the term involving Br on the R.H.S. of the perturbation equation (39). Noting that the basic profiles $\phi_u(z)$ and $\phi_0(x, z)$ are independent of Pr, one may conclude that the relative magnitude of Pr and Br also plays some role leading to the occurrence of the phenomenon. Figures 7–9 reveal that as the value of Ha increases, the phenomenon becomes less appreciable. The effect of the Hartmann number on the asymptotic value of Ra^* is of interest since for the fully developed flow, one has $\partial\phi_0/\partial x = 0$ and the perturbation equations (37) and (39) become identical with those of Chandrasekhar [10] when $Br = 0$. It is found that the present asymptotic results with $Br = 0$ agree with those of [10]. From Figs. 6–9, it is seen that with the increase of Hartmann number, the effect of Brinkman number on the asymptotic value of Ra^* becomes less appreciable. Figures 10 and 11 show clearly the effects of Ha and Pr on the distribution of Ra^* along the axial direction x for given values of Pe and Br. The present investigation shows that the magnetic field has a stabilizing effect and the decreasing Prandtl number

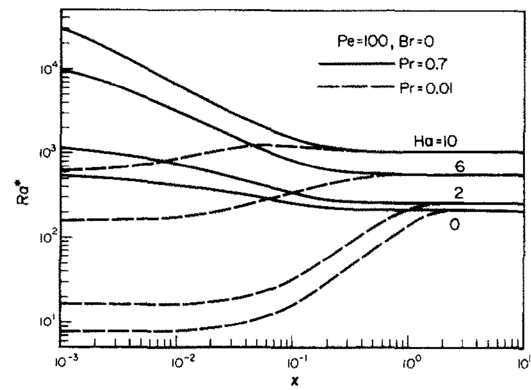

FIG. 10. Hartmann number effect on critical Ra^* in thermal entrance region for $Pe = 100$, $Br = 0$ and $Pr = 0.01, 0.7$.

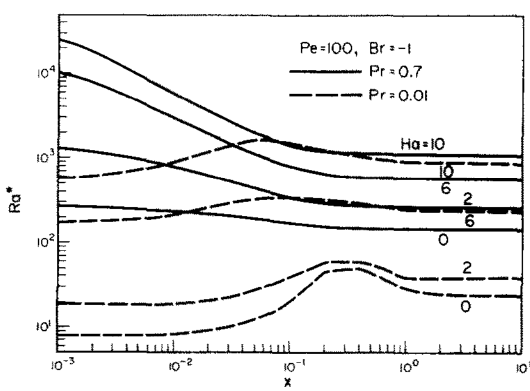

FIG. 11. Hartmann number effect on critical Ra^* in thermal entrance region for $Pe = 100$, $Br = -1$ and $Pr = 0.01, 0.7$.

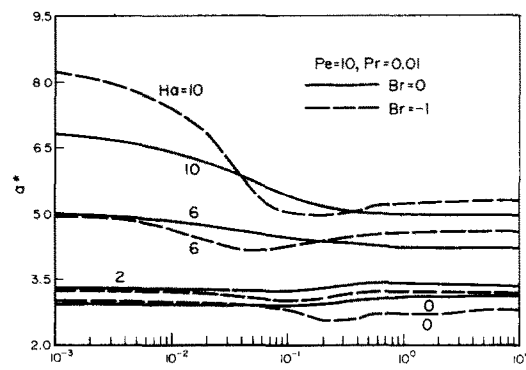

FIG. 12. Hartmann number effect on critical a^* in thermal entrance region for $Pr = 0.01$, $Pe = 10$ and $Br = 0, -1$.

has a destabilizing effect in the thermal entrance region. For reference, the distributions of the wavenumbers a^* are also shown in Fig. 12.

5. CONCLUDING REMARKS

1. The analysis [10] on thermal instability of a horizontal fluid layer confined between two rigid plates subjected to a vertical uniform magnetic field is extended to the case with main flow (Hartmann flow). The present analysis includes the axial conduction, viscous dissipation and Joule heating effects.

2. The numerical results are obtained for $Pr = 0.7$ (air), 0.01 (liquid metal), $Pe = 10, 100, \infty$, $Br = 0, -1$, and $Ha = 0, 2, 6, 10$ with $K = 1$ and $\theta_0 = 1$ only. The case with $K = 1$ signifies the open circuit condition and $\theta_0 = 1$ means $T_0 = T_2$ (entrance temperature is equal to upper plate temperature). At $Br = -1$, the viscous dissipation effect may be considered to be appreciable. It is found that the axial conduction and the magnetic field have a stabilizing effect and the effect of Brinkman number appears to be dependent upon other parameters such as Ha and Pe. It is observed that the combined effect of Prandtl and Brinkman numbers in the perturbation equation (39) may lead to a locally stabilizing effect in some region of the channel before the fully-developed region.

3. For high Prandtl number fluid, the flow is more stable in the thermal entrance region than in the fully-developed region, but the opposite is true for small Prandtl number fluid. However, the Brinkman number has a destabilizing effect in the fully-developed region. When Pr is small, the critical Rayleigh number does not change appreciably throughout the whole entrance length at say $Ha = 10$.

4. The accuracy and convergence of the numerical solution are checked by comparing the present numerical results with those reported in the literature for the limiting cases [7, 10].

5. The present instability results are useful in predicting the onset of longitudinal vortex rolls in wide horizontal rectangular channels and the complete numerical results for Ra^* and a^* are listed in [18].

6. As noted in [3], for low Peclet number flow regime with viscous dissipation effects, the entrance condition of uniform fluid temperature at $\mathbf{x} = 0$ must be regarded as an approximate one. Consequently, numerical calculation is not made for $Pe < 10$.

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INSTABILITE THERMIQUE DE L'ECOULEMENT DE HARTMANN DANS LA REGION D'ETABLISSEMENT DU REGIME THERMIQUE EN CANAL HORIZONTAL LIMITE PAR DES PLAQUES PLANES ET CHAUFFE PAR LE DESSOUS

Résumé—Le déclenchement de l'instabilité sous forme de tourbillons longitudinaux pour l'écoulement laminaire établi de Hartmann dans la région d'établissement du régime thermique en canal horizontal limité par des plaques planes est étudié à l'aide d'une méthode numérique dans le cas d'un champ magnétique vertical uniforme et d'un chauffage par le dessous. Les résultats numériques sont obtenus pour $Pr = 0{,}7$; 0,01; $Pe = 10$; 100; ∞; $Br = 0$; -1; et $Ha = 0{,}2;6{,}10$. On étudie l'influence des nombres de Prandtl, Péclet (conduction axiale), Brinkman (dissipation visqueuse et effet Joule) et Hartmann sur l'instabilité thermique de l'écoulement magnétohydrodynamique.

THERMISCHE INSTABILITÄT DER HARTMANN-STRÖMUNG IM THERMISCHEN EINLAUFBEREICH VON WAAGERECHTEN, VON UNTEN BEHEIZTEN KANÄLEN AUS PARALLELEN PLATTEN

Zusammenfassung—Mit Hilfe einer numerischen Methode wird das Einsetzen der Instabilität in Form von Längswirbein bei voll ausgebildeter Hartmann-Laminarströmung im thermischen Einlaufbereich eines waagerechten Kanals aus parallelen Platten untersucht für den Fall eines einheitlichen senkrechten Magnetfeldes und bei Beheizung von unten. Numerische Ergebnisse wurden erhalten für $Pr = 0{,}7$; 0,01 und $Pe = 10$; 100; ∞ sowie für $Br = 0$; -1 und $Ha = 0$; 2; 6; 10. Die Einflüsse der Prandtl-, Péclet- (Langsleitung), Brinkman- (viskose Dissipation und Joulesche Heizung) und der Hartmann-Zahl auf die thermische Instabilität einer magnetohydrodynamischen Strömung werden untersucht.

ТЕПЛОВАЯ НЕУСТОЙЧИВОСТЬ ТЕЧЕНИЯ ГАРТМАНА В НАЧАЛЬНОМ УЧАСТКЕ ГОРИЗОНТАЛЬНЫХ ПЛОСКОПАРАЛЛЕЛЬНЫХ НАГРЕВАЕМЫХ СНИЗУ КАНАЛОВ

Аннотация — С помощью численного метода исследуется возникновение неустойчивости в виде продольных вихрей при полностью развитом ламинарном течении Гартмана в нагреваемом начальном участке горизонтальных плоскопараллельных каналов при постоянном вертикальном магнитном поле и нагреве снизу. Получены численные результаты для $Pr = 0{,}7$; 0,01; $Pe = 10{,}000$, ∞; $Br = 0$, -1 и $Ha = 0$, 2, 6, 10. Изучается влияние чисел Прандтля, Пекле (осевая проводимость), Бринкмана (вязкая диссипация и нагрев джоулевым теплом) и Гартмана на тепловую неустойчивость магнитогидродинамического потока.